Is converging bounded sequences really harder than converging binary sequences?

Claude Laflamme
University of Calgary

Historical Overview

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section Set Theory & Topology
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Definition (Vojtáș (88))

$$\mathfrak{r} = min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^{\omega}\ \forall b \in 2^{\omega}\ \exists A \in \mathcal{A}\ \lim_{n \in A} b(n) \text{ exists.}\}$$

$$= min\{|\mathcal{A}| : \forall B\ \exists A \in \mathcal{A}\ (A \subseteq^* B \text{ or } A \subseteq^* \omega \setminus B)\}$$

$$= \textit{unsplitting number.}$$

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$$\mathfrak{r}_\sigma = min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega : \ \forall b \in \ell^\infty \ \exists A \in \mathcal{A} \ \lim_{n \in A} b(n) \text{ exists.}\}$$

$$= min\{|\mathcal{A}| : \forall \langle B_n \rangle_n \ \exists A \in \mathcal{A} \ \forall n \ \ (A \subseteq^* B_n \text{ or } A \subseteq^* \omega \setminus B_n)\}$$

$$= \sigma - unsplitting\ number.$$

Proof of the Definition

Proof.

Convergence

$$
\begin{array}{cccccccccc}
 & & & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & \cdots \\
 & & & \| & \| & \| & \| & \| & \| & \cdots \\
 & \chi_0 & = & 0 & 0 & 0 & 1 & 1 & 0 & \cdots \\
\mathfrak{r}_\sigma & \chi_1 & = & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\
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 & \vdots & = & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

□

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			x_0	x_1	x_2	x_3	x_4	x_5	$\cdots$
			$\Vert$	$\Vert$	$\Vert$	$\Vert$	$\Vert$	$\Vert$	$\cdots$
	χ_0	$=$	0	0	0	1	1	0	$\cdots$
$\mathfrak{r}_\sigma$	χ_1	$=$	1	1	0	0	0	0	$\cdots$
	χ_2	$=$	1	1	1	0	0	0	$\cdots$
	$\vdots$	$=$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

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Early Questions

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Question (Vojtáš 89)

Is $\mathfrak{r} = \mathfrak{r}_\sigma$?

Question (Miller 82)

Is $cf(\mathfrak{r})$ uncountable?

Observation

$cf(\mathfrak{r}_\sigma)$ is uncountable.

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For each $X \in [\omega]^\omega$, fix a bijection $\pi_X : \omega \to X$.Now define

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Now given $\langle B_n \rangle_n$, choose:

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Now choose $A \subseteq^* A_n$ for each n and this does unsplit $\langle B_n \rangle_n$.

Theorem (Blass 93)

$$max\{\mathfrak{d}, \mathfrak{r}\} \leq \mathfrak{hom}_n \leq max\{\mathfrak{d}, \mathfrak{r}_\sigma\}.$$

Definition

$$\mathfrak{d} = min\{|\mathcal{D}| : \mathcal{D} \subseteq \omega^\omega \ \forall g \in \omega^\omega; \exists f \in \mathcal{D} \ f \geq^* g.\}$$

$$\mathfrak{hom}_n = min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega : \ \forall h : [\omega]^n \to 2 \ \exists A \in \mathcal{A} \ h \upharpoonright [A]^n =^* cte.\}$$

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Proof $\mathfrak{hom}_2 \leq max\{\mathfrak{d}, \mathfrak{r}_\sigma\}$.

For h in a dominating family $\mathcal{D}$, X in a σ-unspliting family $\mathcal{R}$ and $Y \in \pi_X(\mathcal{R})$, choose

$$H(h, X, Y) \subseteq Y \text{ infinite so that } x < y \implies h(x) < y$$

. Then $\{H(h, X, Y) : h \in \mathcal{D}, X \in \mathcal{R}, Y \in \pi_x(\mathcal{R})\}$ works for $\mathfrak{hom}_2$.

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If $h \restriction [A]^2 = cte$, then $\chi_n \restriction A =^* cte$ for all n . □

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Given $\alpha, \gamma < \kappa$, construct $C_{\alpha,\gamma} \in [\omega]^\omega$, recursively as follows:

$$\begin{aligned} C^0_{\alpha,\gamma} &= \omega \\ C^{n+1}_{\alpha,\gamma} &= \begin{cases} A_{\pi_\alpha(g_\gamma(n))} & \text{if this set is almost contained in } C^n_{\alpha,\gamma}, \\ C^n_{\alpha,\gamma} & \text{otherwise.} \end{cases} \end{aligned}$$

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In the end, let $C_{\alpha,\gamma}$ be an infinite pseudointersection of the $C^n_{\alpha,\gamma}$.

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In the end, let $C_{\alpha,\gamma}$ be an infinite pseudointersection of the $C^n_{\alpha,\gamma}$.
Show that the sets $C_{\alpha,\gamma}$ form a σ-unsplitting family: given $\langle D_n \rangle_n$:

$$E = \{F \subset \mathfrak{r} : \forall n \forall \beta \in F \exists \delta \in F\ A_\delta \subseteq^* A_\beta \cap D_n \text{ or } A_\delta \subseteq^* A_\beta \setminus D_n\}$$

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Corollary (Brendle Just (00))

If $\mathfrak{r} < \mathfrak{r}_\sigma$, *then*

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- $\mathfrak{d} < \mathfrak{r}_\sigma$ *in random real model.*

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- $cf([\mathfrak{r}]^{\aleph_0}) > \mathfrak{r} \implies 2^\omega \geq \mathfrak{r} \geq \aleph_\omega$.
- $\mathfrak{d} < \mathfrak{r}_\sigma$ *in random real model.*
- $\mathfrak{r} < \mathfrak{u}$ *in Goldstern-Shelah model.*
- *Finite support iteration forces* $non(\mathcal{M}) \leq \mathfrak{r}$, *so cannot yield* $\mathfrak{r} = \aleph_1 < \mathfrak{r}_\sigma = \aleph_2$.

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- Thus $\mathfrak{r} = \mathfrak{r}_\sigma = \aleph_1$.

□

Remark

Countable support iteration of proper posets over CH cannot yield $\mathfrak{r} < \mathfrak{r}_\sigma$.

Proof.

- Wlog $V \cap [\omega]^\omega$ is an unsplitting family.
- For each $\alpha < \omega_2$ V contains an ultrafilter generating an ultrafilter in $V[G_\alpha]$.
- $\implies V \cap [\omega]^\omega$ is a $\sigma-$unsplitting family.
- Thus $\mathfrak{r} = \mathfrak{r}_\sigma = \aleph_1$.

□

Conjecture

Andrzej will show uncountable support iteraition won't work either....

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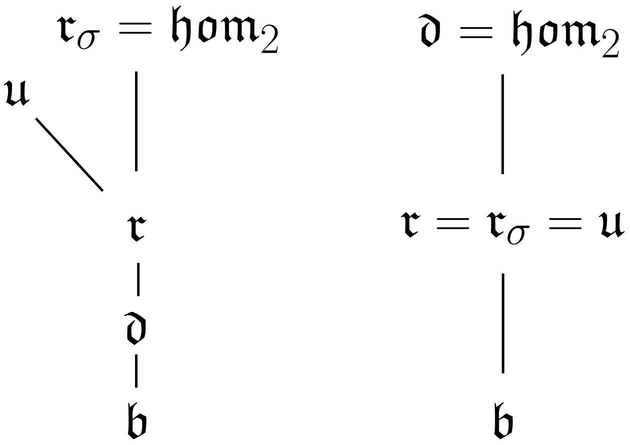
- Hence $\mathfrak{r} = \mathfrak{u}$.
- So the ultrafilter is a P-point ($\mathfrak{u} = \mathfrak{r} < \mathfrak{d}$), thus $\mathfrak{r} = \mathfrak{r}_\sigma$.

□



$\mathfrak{r} \geq \mathfrak{d}$ $\qquad\qquad$ $\mathfrak{r} < \mathfrak{d}$

$\mathfrak{fr}$ and $\mathfrak{fr}_\sigma$

Definition (Brendle 98)

$$
\begin{aligned}
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$\mathit{min}\{\mathfrak{d}, \mathfrak{r}\} = \mathit{min}\{\mathfrak{d}, \mathfrak{r}_\sigma\}$ and thus :

$$\mathfrak{fr} = \mathfrak{fr}_\sigma$$

Question

If $\mathfrak{r} = \aleph_1$, *is* $\mathfrak{r} = \mathfrak{r}_\sigma$*?*

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Conjecture

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